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A Class of Asymptotically Efficient and Consistent Sequential Procedures to Construct Fixed-size Confidence Regions

Ajit Chaturvedi, S.K. Pandey[1] and Sheela Misra[1]
Department of Statistics, C.C.S. University, Meerut-250005
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SUMMARY

A class of sequential procedures is developed for constructing confidence regions of pre-assigned width and coverage probability for the parameter(s) (scalar or vector-valued) of a population in the presence of nuisance parameter. The proposed class is shown to be 'asymptotically efficient and consistent' in Chow-Robbins (1965) sense. By means of various examples, it is shown that many estimation problems can be tackled by the proposed class.

Key-words : Fixed-size, Confidence regions, Sequential estimation, Stopping time, Asymptotic efficiency, Consistency.

1. *Introduction and the Fixed-Sample Size Procedure*

Sequential procedures to construct confidence regions of pre-assigned width and coverage probability for the parameter(s) of various distributions have been considered by many authors. For a brief review on the literature, one may refer to Govindarajulu [15], [16]. In the present note, exploiting the common distributional properties of the estimators of the parameter(s) of interest and those of nuisance parameters (which motivate one to adopt sequential procedures) under different models, a class of sequential procedures is developed. The proposed class is shown to be 'asymptotically efficient and consistent' in Chow-Robbins [10] sense. The class and its properties are presented in Section 2. In Section 3, by means of various examples, we illustrate that many estimation problems can be handled with the help of the proposed class. The set-up of the problem can be described as follows:

Let us consider a sequence $\{X_i\}$, $i = 1, 2, \ldots$ of iid rv's from a t-variate $(t \geq 1)$ absolutely continuous population $f(\mathbf{x}; \theta, \psi)$, where θ is a $t \times 1$ vector of unknown parameter(s) of interest and ψ is a nuisance parameter. Denoting

1 Department of Statistics, Lucknow University, Lucknow-226007, India.

by R^t and R^+, the t-dimensional Euclidean space and the positive-half of the real line, respectively, let $(\theta', \psi)' \in R^t \times R^+$. Having recorded a random sample $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ of size $n \geq t+1$, let $\hat{\theta}(\mathbf{X}_1, ..., \mathbf{X}_n) \equiv \hat{\theta}_n$ and $\psi(\mathbf{X}_1, ..., \mathbf{X}_n) \equiv \hat{\psi}_n$ be the estimators of θ and ψ, respectively. We make the following assumptions.

(a_1): There exist a known $t \times t$ positive definite matrix Q, a number $\delta \in (0, 1]$ and an integer r (≥ 1) such that

$$n\,\psi^{-1}\,[(\hat{\theta}_n - \theta)'\,Q\,(\hat{\theta}_n - \theta)]^{\delta} \sim \chi^2_{(r)}$$

where $\chi^2_{(r)}$ denotes a chi-square variate with r degrees of freedom.

(a_2): For all $n \geq t+1$, $\hat{\theta}_n$ and $\hat{\psi}_n$ are stochastically independent.

(a_3): There exist integers p (≥ 1) and q (≥ 1) such that for $n \geq q+1$,

$$p\,(n-q)\,\hat{\psi}_n / \psi = \sum_{j=1}^{n-q} Z_j^{(p)}, \qquad \text{with } Z_j^{(p)} \sim \chi^2_{(p)}$$

(a_4): $\hat{\psi}_n \xrightarrow{a.s.} \psi$ as $n \longrightarrow \infty$

For specified $d \in (0, \infty)$ and $\alpha \in (0, 1)$, suppose one wishes to construct a confidence region R_n for θ in R^t such that the width of R_n is bounded by 2d and $P(\theta \in R_n) \geq \alpha$. We define

$$R_n = [\mathbf{Z}: \{(\hat{\theta}_n - \mathbf{Z})'Q\,(\hat{\theta}_n - \mathbf{Z})\}^{\delta} \leq d^2] \tag{1.1}$$

Denoting by $F^{(r)}(.)$, the cdf of a $\psi^2_{(r)}$ rv and utilizing (a_1), we obtain from (1.1),

$$P(\theta \in R_n) = P[\{(\hat{\theta}_n - \hat{\theta})'\;Q\,(\hat{\theta}_n - \theta)\}^{\delta} \leq d^2]$$
$$= F^{(r)}(n\,\psi^{-1}\,d^2)$$

Let a^2 be the upper 100α% point of $\chi^2_{(r)}$ distribution, i.e.

$$F^{(r)}(a^2) = \alpha$$

Using monotonicity property of cdf, we conclude from (1.2) and (1.3) that for known ψ, in order to achieve $P(\theta \in R_n) \geq \alpha$, the sample size required is the smallest positive integer $n \geq n_o$,

where $\quad n_o = (a/d)^2\,\psi$

However, in the absence of any knowledge about ψ, no fixed sample size procedure achieves the goals of fixed-size and pre-assigned coverage probability simultaneously for all values of ψ. To meet the requirements, we propose a class $\mathcal{J}$ of sequential procedures, which is discussed in the following section.

2. *The Class $\mathcal{J}$ of Sequential Procedures and its Properties*

Let us start with a sample of size $m \geq \max\{t+1, q+1\}$. Then, motivated by (1.4), the stopping time $N \equiv N(d)$ is the smallest positive integer $n \geq m$ such that

$$n \geq (a/d)^2 \hat{\psi}_n \tag{2.1}$$

After stopping, we construct the region

$$R_N = [\mathbf{Z}: \{(\hat{\theta}_N - \mathbf{Z})' \, Q \, (\hat{\theta}_N - \mathbf{Z})\}^{\delta} \leq d^2] \quad \text{for} \quad \theta$$

Now we prove the following theorem, which establishes the results that the class $\mathcal{J}$ of sequential procedures defined above is 'asymptotically efficient and consistent' in Chow-Robbins [10] sense. In what follows, we denote by $V_n = p(n-q)\hat{\psi}_n / \psi$.

Theorem: N is well-defined stopping rule. (2.2)

$$\lim_{d \to o} N = \infty \text{ a.s.} \tag{2.3}$$

$$\lim_{d \to o} (N / n_o) = 1 \text{ a.s.} \tag{2.4}$$

$$E(N) \leq n_o + m - 1 \tag{2.5}$$

$$\lim_{d \to o} E(N / n_o) = 1 \tag{2.6}$$

$$\lim_{d \to o} P(\theta \in R_N) = \alpha \tag{2.7}$$

We omit the proof which follows using standard techniques.

3. *Estimation Problems having Solutions Provided by the Class $\mathcal{J}$*

3.1 Estimation of the Mean Vector of a Multinormal Population

Let $\{\mathbf{X}_i\}$, i = 1, 2, . . . be a sequence of iid rv's from a p-variate normal population $N_p(\mathbf{x}: \mu, \sigma^2\Sigma)$, where μ is the $p \times 1$ unknown mean vector, $\sigma^2 \in (0, \infty)$ is an unknown scalar and Σ is a known $p \times p$ positive definite matrix.

Suppose, one wishes to construct an ellipsoidal confidence region R_n for μ, such that the width R_n is bounded by 2d and $P(\mu \in R_n) \geq \alpha$. Having recorded $X_1, \ldots, X_n, n \geq p+1$, we use $\overline{X}_n = n^{-1} \sum_{i=1}^{n} X_i$ and $\hat{\sigma}_n^2 = [p(n-1)]^{-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)' \Sigma^{-1} (X_i - \overline{X}_n)$ as the estimators of μ and σ^2, respectively. Thus, $t = p$, $\theta = \mu$, $\psi = \sigma^2$, $\hat{\theta}_n = \overline{X}_n$ and $\hat{\psi}_n = \hat{\sigma}_n^2$. It can be seen [see, Wang [32]] that $(a_1)-(a_4)$ are satisfied for $r = p$, $s = 1$. We construct

$$R_N = [(\overline{X}_N - Z)' \Sigma^{-1} (\overline{X}_N - Z) \leq d^2] \text{ for } \mu$$

For sequential procedures to construct fixed-size confidence regions for the mean vector of a multinormal population, one may refer to Srivastava [23], Srivastava and Bhargava ([26], [28]), Jones [18] and Singh and Chaturvedi [21].

3.2 Estimation of the Regression Parameters in a Linear Model

Let us consider the linear model $Y_n = X_n \beta + \varepsilon_n$, Y_n is an observed $n \times 1$ random vector, X_n is a $n \times p$ matrix of rank p, β is a $p \times 1$ vector of unknown parameters and ε_n is the disturbance term following $N_n(0, \sigma^2 I_n)$ distribution. We have to construct an ellipsoidal confidence region R_n for β such that the width of R_n is bounded by 2d and $P(\beta \in R_n) \geq \alpha$. The ordinary least-squares estimator of β is $\hat{\beta}_n = (X_n' X_n)^{-1} X_n' Y_n$ and we use $\hat{\sigma}_n^2 = (n-p)^{-1} Y_n' [I_n - X_n (X_n' X_n)^{-1} X_n'] Y_n$ to estimate σ^2. Thus, $t = p$, $\theta = \beta$, $\psi = \sigma^2$, $\hat{\theta}_n = \hat{\beta}_n$ and $\hat{\psi}_n = \hat{\sigma}_n^2$. It is easy to verify [see, Chaturvedi, [7]] that $(a_1)-(a_4)$ hold for $p = 1$, $r = q = p$. We construct

$$R_N = [(\hat{\beta}_N - Z)' (X_N' X_N) (\hat{\beta}_N - Z) \leq d^2] \text{ for } \beta$$

Sequential procedures to construct fixed-size confidence regions for regression parameters in a linear model have been developed and studied by Wijsman [33], Gleser ([13], [14]), Srivastava ([23], [24]) and Chaturvedi [8].

3.3 Comparisons of a Control Against Several Treatments

Let us consider the problem of obtaining simultaneous confidence intervals for the differences $\delta_i = \mu_0 - \mu_i$, $i = 1, 2, \ldots, k$, where μ_0 and μ_i are, respectively, the means of the control treatment and the i^{th} treatment. Let us denote by $\{X_{ij}\}$, $j = 1, 2, \ldots$, the j^{th} observation on the i^{th} treatment. From these observations, we can construct vectors $X_1, X_2, \ldots$ where $X_j' = (X_{0j} - X_{1j}, X_{0j} - X_{2j}, \ldots, X_{0j} - X_{kj})$. These vectors are iid normal random variables with mean vector $\mu' = (\delta_1, \delta_2, \ldots, \delta_k)$ and the covariance matrix $\sigma^2 V$, where $\sigma^2 = \mathrm{Var}(X_{0j} - X_{1j})$ is unknown and

$$V = \begin{bmatrix} 1 & 1/2 & \ldots & 1/2 \\ 1/2 & 1 & \ldots & 1/2 \\ \ldots & \ldots & \ldots & \ldots \\ 1/2 & 1/2 & \ldots & 1 \end{bmatrix}$$

We have to construct a confidence interval R_n of width $2d_i$ for δ_i, such that $P(\delta_i \in R_n) \geq \alpha$. Having observed a random sample $X_{i1}, \ldots, X_{in}$ of size $n \geq 2$, for $\overline{X}_{i(n)} = n^{-1} \sum_{j=1}^{n} X_{ij}$, we use $\hat{\delta}_i(n) = \overline{X}_{0(n)} - \overline{X}_{1(n)}$ and $S_n^2 = [k(n-1)]^{-1} \sum_{j=1}^{n} (X_j - \overline{X}_j)' V^{-1} (X_j - \overline{X}_j)$ as the estimators of δ_i and σ^2, respectively. Thus, $t = 1$, $\theta = \delta_i$, $\psi = \sigma^2$, $\hat{\theta}_n = \hat{\delta}_{i(n)}$ and $\hat{\psi}_n = S_n^2$. It can be verified that $(a_1) - (a_4)$ hold for $\delta = Q = I_{1\times 1} = 1, r = p = k$ and $q = 1$. We construct $R_N = [\hat{\delta}_{i(N)} \pm d_i]$ for δ_i.

Fixed sample size and sequential procedures for this estimation problem have been proposed and studied by Dunnett [11] and Jones [18], respectively.

3.4 Estimation of the Mean of Inverse Gaussian Distribution with Prescribed Proportional Closeness

Let $\{X_i\}$, $i = 1, 2, \ldots$ be a sequence of iid rv's from an inverse Gaussian population

$$f(x; \mu, \lambda) = \left(\frac{\lambda}{2\pi x^9}\right)^{1/2} \exp\left[-\frac{\lambda (x-\mu)^2}{2\mu^2 x}\right]; \quad x > 0$$

where both the parameters $\mu \in (0, \infty)$ and $\lambda \in (0, \infty)$ are unknown. Having observed a random sample of $X_1, \ldots, X_n$ of size $n (\geq 2)$, let us define $\overline{X}_n = n^{-1} \sum_{i=1}^{n} X_i$ and $\hat{\lambda}_n = (n-1)^{-1} \sum_{i=1}^{n} (X_i^{-1} - \overline{X}_n^{-1})$ as the estimators of μ and

λ^{-1}, respectively. Consider the loss in estimating μ by $\overline{X}_n$ to be a zero-one loss function given by

$$L(\mu, \overline{X}_n) = \begin{cases} 1, & \text{if } \dfrac{|\overline{X}_n - \mu|}{\sqrt{\overline{X}_n}} > d\,\mu \\ 0, & \text{otherwise} \end{cases}$$

Our target is to achieve $E[L(\mu, \overline{X}_n)] \le \alpha$. Thus, $t = 1$, $\theta = \mu$, $\psi = \lambda^{-1}$, $\hat{\theta}_n = \overline{X}_n$ and $\hat{\psi}_n = \hat{\lambda}_n$. It can be verified [see Chaturvedi [5]] that $(a_1) - (a_4)$ are satisfied for $\delta = r = p = q = 1$ and $Q = (\mu^2 \overline{X}_n)^{-1} I_{1\times 1} = (\mu^2 \overline{X}_n)^{-1}$.

Sequential procedures for this estimation problem been developed and studied by Chaturvedi [5], [6]. For some related work, one may also refer to Singh and Chaturvedi [22].

3.5 Estimation Problems Related to Multiple Comparison Procedures

We now consider estimation problems which take place in multiple comparison procedures. For a detailed discussion on this topic, one may refer to Hochberg and Tamhane [17]. We take the linear model same as that defined in Section 3.2. In many problems of multiple comparisons, one may be interested in estimating the parametric functions of the components of a $k \times 1$ $(1 \le k \le p)$ subvector θ of β. Here θ may be the effects of a certain qualitative factor (or a combination of two or more qualitative factors), which may be referred to as the treatment factor and these $k\,(\ge 2)$ levels as the treatments of main interest. It may also contain the effects of factors such as blocks and covariates, included to account for the variability among the experimental (or the observational) units and thus yield more precise comparisons among the treatment effects.

Let $\hat{\theta}_n$ be the corresponding subvector of $\hat{\beta}_n$ and V be the $k \times k$ submatrix of $(X'_n X_n)^{-1}$ corresponding to the $\hat{\theta}_n$ part of $\hat{\beta}_n$. Let $L = (l_1, \ldots, l_p)'$ be a known matrix. Consider a set of parametric functions $\gamma_i = l'_i\,\theta$ and write $\gamma = (l'_1\,\theta, \ldots, l'_p\,\theta)'$. Let $\hat{\gamma}_n = L\,\hat{\theta}_n$ be the least-squares estimator of γ. In order to estimate σ^2, we use $S_n^2 = v^{-1} \| Y - \hat{Y}_n \|^2$, where $\hat{Y}_n = X_n \hat{\beta}_n$ and $v = n - s_0$ is the error degrees of freedom and $s_0 = \text{rank}(X_n)$. Our goal is to construct an ellipsoidal confidence region R_n for γ of diameter bounded by 2d and confidence coefficient atleast α. It can be verified that $(a_1) - (a_4)$ hold for $Q = n^{-1}(LVL')^{-1}$, $\delta = 1$, $r = p$ and $q = s_0$. We propose the region

$$R_N = [\mathbf{Z}: (\hat{\gamma}_N - \mathbf{Z})' (LVL')^{-1} (\hat{\gamma}_N - \mathbf{Z}) \leq d^2] \qquad \text{for } \gamma$$

3.6 Estimation of the Mean of Intra-class Model

Let us consider the intra-class model $X_i = \mu + \varepsilon_i$, $i = 1, 2, \ldots, n$, where ε_i's are normally distributed with $E(\varepsilon_i) = 0$ and

$$\text{Cov}(\varepsilon_i, \varepsilon_j) = \begin{cases} \sigma^2, & \text{if } i = j \\ \rho\,\sigma^2, & \text{if } i \neq j;\ i, j = 1, 2, \ldots, n \end{cases}$$

The parameters $\mu \in (-\infty, \infty)$, $\sigma \in (0, \infty)$ and $\rho \in (-1, 0)$ are unknown. We have to construct a confidence interval of width 2d and coverage probability at least α for μ. We use $\overline{X}_n = n^{-1} \sum_{i=1}^{n} X_i$ and $S_n^2 = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$ as the estimators of μ and $\sigma^2(1-\rho)$, respectively. Thus, $t = 1$, $\theta = \mu$, $\psi = \sigma^2(1-\rho)$, $\hat{\theta}_n = \overline{X}_n$ and $\hat{\psi}_n = S_n^2$. We note that $(a_1) - (a_4)$ are satisfied for $\delta = r = p = q = I_{1 \times 1} = 1$. We construct the interval $R_N = [\overline{X}_N \pm d]$ for μ.

3.7 Estimation of the Scale Parameter of Pareto Distribution

Let us consider a sequence $\{X_i\}$, $i = 1, 2, \ldots$ of iid rv's from the first kind of Pareto distribution

$$f(x; \theta, \sigma) = \sigma^{-1} \theta^{-1/\sigma} x^{-1/\sigma - 1};\ x \geq \theta > 0,\ \sigma > 0$$

Both, the scale parameter θ and the shape parameter σ are unknown and we have to construct a confidence interval R_n of width bounded by 2d for $\ln\theta$ such that $P(\ln\theta \in R_n) \geq \alpha$. Having recorded a random sample $X_1, \ldots, X_n$ of size $n(\geq 2)$, for $X_{n(1)} = \min(X_1, \ldots, X_n)$, we use $u_{n(1)} = \ln X_{n(1)}$ and $\hat{\sigma}_n = (n-1)^{-1} \sum_{i=1}^{n} \ln(X_i / X_{n(1)})$ as the estimators of $\ln\theta$ and σ, respectively. Thus, $\theta = \ln\theta$, $\psi = \sigma$, $\hat{\theta}_n = u_{n(1)}$ and $\hat{\psi}_n = \hat{\sigma}_n$. We note that $(a_1) - (a_4)$ are satisfied for $\delta = 1/2$, $Q = I_{1 \times 1} = 1$, $r = p = 2$ and $q = 1$. We propose the interval $R_N = [u_{N(1)} \pm d]$ for $\ln\theta$.

For sequential procedure to estimate the scale parameter of Pareto distribution, one may refer to Wang [31].

3.8 Estimation of the Location Parameter of Exponential Distribution

Let $\{X_i\}$, $i = 1, 2, \ldots$ be a sequence of iid rv's from negative exponential distribution

$$f(x; \mu, \sigma) = \sigma^{-1} \exp[-(x-\mu)/\sigma] \; I(x > \mu)$$

where $\mu \in (-\infty, \infty)$ and $\sigma \in (0, \infty)$ are the unknown parameters. Suppose, we have to construct a confidence interval R_n of width bounded by 2d for μ, such that $P(\mu \in R_n) \geq \alpha$. Having observed $X_1, \ldots, X_n$, $n \geq 2$, we use $X_{n(1)} = \min(X_1, \ldots, X_n)$, and $\hat{\sigma}_n = (n-1)^{-1} \sum_{i=1}^{n} \ln(X_i - X_{n(1)})$ as the estimators of μ and σ, respectively. Here $t = 1$, $\theta = \mu$, $\psi = \sigma$, $\hat{\theta}_n = X_{n(1)}$ and $\hat{\psi}_n = \hat{\sigma}_n$. It is easy to see that $(a_1)-(a_4)$ are satisfied for $\delta = 1/2$, $Q = I_{1\times 1} = 1$, $r = p = 2$ and $q = 1$. We propose the interval $R_N = [X_{N(1)} \pm d]$ for μ.

Sequential procedures to construct fixed-width confidence interval for the location parameter of a negative exponential distribution have been developed and studied by Basu [4], Swanepoel and van Wyk [30] and Chaturvedi and Shukla [9].

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